J. DIFFERENTIAL GEOMETRY
11 (1976) 467- 474

ORTHONORMAL FRAMES ON 3-DIMENSIONAL RIEMANNIAN MANIFOLDS

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1. Introduction

Let (M, g) be a Riemannian manifold. An orthonormal frame $(X_i, i = 1, 2, \cdots, m = \dim M)$ on an open set U of M is called a Killing frame if each X_i is a Killing vector field on $U = (U, g = g \mid U)$. D'Atri and Nickerson [1] proved that if (X_i) is a Killing frame on (U, g), then (U, g) is locally symmetric. It is also proved that such a space (U, g) is of nonnegative curvature.

In this paper we prove

Theorem A. *If a 3-dimensional Riemannian manifold (M, g) admits a Killing frame (X_i) on an open set U, then (U, g) is of nonnegative constant curvature.*

Next we study orthonormal frames satisfying some additional conditions.

Theorem B. *Let (M, g) be a 3-dimensional Riemannian manifold, and (X_i) an orthonormal frame on an open set U of M such that*

$$(1) \qquad [X_1, X_2] = aX_3\ , \quad [X_2, X_3] = aX_1\ , \quad [X_3, X_1] = aX_2$$

for some constant a. Then (X_i) is a Killing frame, and (U, g) is of constant curvature $\frac{1}{4}a^2$.

Theorem B for the case $a \neq 0$ follows from the next more general Theorem B*.

Theorem B*. *Let (M, g) be a 3-dimensional Riemannian manifold, and (X_i) an orthonormal frame on an open set U of M such that*

$$(2) \qquad [X_1, X_2] = cX_3\ , \quad [X_2, X_3] = aX_1\ , \quad [X_3, X_1] = bX_2$$

for some positive (or negative) constants a, b, c. Let θ and ψ be the 1-forms on U which are the duals of X_1 and X_2 with respect to g. Then U admits a Riemannian metric $\bar{g}$ of constant curvature 1 such that

$$(3) \qquad g \mid U = \frac{4}{ab}\bar{g} + \frac{a-c}{c}\theta \otimes \theta + \frac{b-c}{b}\psi \otimes \psi\ .$$

Theorem C. *Let (M, g) be a 3-dimensional Riemannian manifold, and (X_i)*

Communicated by K. Yano, December 27, 1974.

an orthonormal frame on U such that

$$[X_1, X_2] = 0\,, \quad [X_2, X_3] = aX_1\,, \quad [X_3, X_1] = aX_2 \tag{4}$$

for some constant a. Then X_3 is a Killing vector field on U, and (U, g) is locally flat.

Theorem C*. *Let (M, g) be a 3-dimensional Riemannian manifold, and (X_i) an orthonormal frame on U such that*

$$[X_1, X_2] = 0\,, \quad [X_2, X_3] = aX_1\,, \quad [X_3, X_1] = bX_2 \tag{5}$$

for some positive (or negative) constants a, b. Let θ be the 1-form dual to X_1 with respect to g. Then U admits a flat metric $\bar{g}$ such that

$$g\,|\,U = \bar{g} + \frac{a-b}{a}\theta \otimes \theta\,. \tag{6}$$

An interesting application of Theorem B* is given on the tangent sphere bundles of a 2-dimensional Riemannian manifold of constant curvature.

Theorem D. *Let (M, g) be a 2-dimensional oriented Riemannian manifold of constant curvature $K > 0$, and (T_uM, g^S) the tangent sphere bundle (consisting of tangent vectors of length u) with the induced metric from the Sasaki metric g^S of the tangent bundle TM. Let J be the natural almost complex structure tensor on M. Then J^* and J^v (defined by (24) and (25)) are vector fields on TM which are tangent to each tangent sphere bundle T_uM. Let F be the geodesic flow vector field, which is also tangent to T_uM. Then on (T_uM, g^S) we have the global orthonormal frame $(X_1 = J^v/u,\ X_2 = F/u,\ X_3 = J^*/u)$ which satisfies (2) with $a = Ku$, $b = c = 1/u$. Therefore T_uM admits a Riemannian metric $\bar{g}$ of constant curvature 1 such that*

$$g^S\,|\,T_uM = \frac{4}{K}\bar{g} + \frac{Ku^2-1}{Ku^2}\theta \otimes \theta\,, \tag{7}$$

where θ is the 1-form dual to J^v/u on (T_uM, g^S).

In particular, (T_uM, g^S) with $u^2 = 1/K$ is of constant curvature $\frac{1}{4}K$.

As a corollary, if we put $K = 1$ and $u = 1$, we have a theorem of Klingenberg and Sasaki [2] that the tangent unit sphere bundle of a 2-dimensional sphere of constant curvature 1 is a real projective 3-space of constant curvature $\frac{1}{4}$.

2. Proofs of Theorems A, B, B*, C and C*

Denote by ∇ the Riemannian connection of (M, g), and by R the Riemannian curvature tensor

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z ,$$

where X, Y, Z are vector fields on M. For our purpose the following lemma is useful.

Lemma 2.1 (*D'Atri and Nickerson* [1, *proof of Lemma* 3.4]). *For a Killing frame* (X_i) *on* U *we have*

$$4R(X_k, X_l)X_j = -[[X_k, X_l], X_j] . \tag{8}$$

From now on in this section we assume that (M, g) is a 3-dimensional Riemannian manifold and $(U, g = g\,|\,U)$ is an open set where an orthonormal frame $(X_i,\ i = 1, 2, 3)$ is defined.

Lemma 2.2. *Assume that* $[X_1, X_2] = cX_3$, $[X_2, X_3] = aX_1$, $[X_3, X_1] = bX_2$ *hold for some positive* (*or negative*) *constants* a, b, c. *Let* θ *and* ψ *be the duals of* X_1 *and* X_2 *with respect to* g. *If we put* $s = a/|a| = \pm 1$ *and*

$$\bar{X}_1 = \frac{2s}{\sqrt{bc}} X_1 , \quad \bar{X}_2 = \frac{2s}{\sqrt{ac}} X_2 , \quad \bar{X}_3 = \frac{2s}{\sqrt{ab}} X_3 , \tag{9}$$

$$4\bar{g} = abg + b(c - a)\theta \otimes \theta + a(c - b)\psi \otimes \psi , \tag{10}$$

then we have a new orthonormal frame $(\bar{X}_i)$ *with respect to* $\bar{g}$ *such that*

$$[\bar{X}_1, \bar{X}_2] = 2\bar{X}_3 , \quad [\bar{X}_2, \bar{X}_3] = 2\bar{X}_1 , \quad [\bar{X}_3, \bar{X}_1] = 2\bar{X}_2 . \tag{11}$$

Proof. By the assumption on a, b, c, the tensor $\bar{g}$ defined by (10) is a Riemannian metric on U. Then it is easy to verify the required relations.

Lemma 2.3. *If the relations*

$$[X_1, X_2] = aX_3 , \quad [X_2, X_3] = aX_1 , \quad [X_3, X_1] = aX_2 \tag{12}$$

hold on U *for some constant* a, *then each* X_i *is a Killing vector field on* U.

Proof. It is easy to verify that $L_{X_i} g = 0$ by the following relations:

$$\begin{aligned} 0 &= L_{X_i}(g(X_j, X_k)) \\ &= (L_{X_i} g)(X_j, X_k) + g([X_i, X_j], X_k) + g(X_j, [X_i, X_k]) , \end{aligned}$$

where L_{X_i} denotes the Lie derivation with respect to X_i.

Lemma 2.4. *Let* (X_i) *be a Killing frame on* U. *Then we have a constant* a *such that* (12) *holds.*

Proof. Since X_i, $i = 1, 2, 3$, are orthonormal Killing vector fields, we have $g(L_{X_1} X_2, X_1) = 0 = g(L_{X_1} X_2, X_2)$, that is, $[X_1, X_2] = cX_3$ for some function c on U. However, since X_3 and cX_3 are both Killing vector fields, c must be constant by a classical result. Similarly, $[X_2, X_3] = aX_1$ and $[X_3, X_1] = bX_2$ hold for some constants a and b. By

$$0 = (L_{X_1}g)(X_2, X_3) = g([X_1, X_2], X_3) + g(X_2, [X_1, X_3]) ,$$

we have $b = c$. Similarly, $a = b$. Hence Lemma 2.4 is proved.

Proof of Theorem A. Let (X_i) be a Killing frame on U. Then (X_i) satisfies (12) by Lemma 2.4. Using Lemma 2.1 we get

$$4g(R(X_1, X_2)X_2, X_1) = -g([aX_3, X_2], X_1) = a^2 .$$

Changing indices we see that (U, g) is of constant curvature $\frac{1}{4}a^2$. Here we note that (1) is invariant under the rotation $(X_i) \to (\alpha_i^r X_r)$, $\alpha \in SO(3)$.

Proof of Theorem B. This follows from Lemma 2.3 and Theorem A.

*Proof of Theorem B**. For a given orthonormal frame (X_i) on U, we define a new metric $\bar{g}$ and a new orthonormal frame $(\bar{X}_i)$ with respect to $\bar{g}$ by (10) and (9) in Lemma 2.2. Then the relations (11) hold. By Theorem B, $\bar{g}$ is of constant curvature 1. Hence (3) and (10) are equivalent.

Lemma 2.5. *Assume that (X_i) satisfies*

$$[X_1, X_2] = 0 , \quad [X_2, X_3] = aX_1 , \quad [X_3, X_1] = bX_2$$

for some positive(or negative) constants a, b. Let θ be the dual of X_1 with respect to g. If we put $s = a/|a| = \pm 1$ and

$$\bar{X}_1 = s\sqrt{a/b}X_1 , \quad \bar{X}_2 = sX_2 , \quad \bar{X}_3 = sX_3 , \tag{13}$$

$$\bar{g} = g + \frac{b-a}{a}\theta \otimes \theta , \tag{14}$$

then $(\bar{X}_i)$ is an orthonormal frame such that

$$[\bar{X}_1, \bar{X}_2] = 0 , \quad [\bar{X}_2, \bar{X}_3] = \sqrt{ab}\bar{X}_1 , \quad [\bar{X}_3, \bar{X}_1] = \sqrt{ab}\bar{X}_2 . \tag{15}$$

Proof. It can be proved by a simple calculation.

Lemma 2.6. *Assume that (X_i) satisfies*

$$[X_1, X_2] = 0 , \quad [X_2, X_3] = aX_1 , \quad [X_3, X_1] = aX_2 \tag{16}$$

for some constant $a \neq 0$.

(i) *X_3 is a Killing vector field on U. By ϕ_t we denote the local 1-parameter group of local isometries generated by X_3.*

(ii) *The distribution defined by X_1 and X_2 is completely integrable. Let N be an integral submanifold such that $W = (\phi_t N, |t| < \varepsilon) \subset U$, and define vector fields X_1^* and X_2^* on W by*

$$(X_1^*)_{\phi_t x} = \cos at\, (X_1)_{\phi_t x} - \sin at\, (X_2)_{\phi_t x} , \tag{17}$$

$$(X_2^*)_{\phi_t x} = \sin at\, (X_1)_{\phi_t x} + \cos at\, (X_2)_{\phi_t x} , \tag{18}$$

for $x \in N$ *and* $\phi_t x \in W$. *Then* $(X_1^*, X_2^*, X_3^* = X_3)$ *is an orthonormal frame with respect to* g *such that*

$$(19) \qquad [X_i^*, X_j^*] = 0 , \qquad i, j = 1, 2, 3 .$$

Proof. By (16) we can verify that $(L_{X_3}g)(X_i, X_j) = g([X_3, X_i], X_j) + g(X_i, [X_3, X_j]) = 0$, so that X_3 is a Killing vector field. From $[X_1, X_2] = 0$ it follows that the distribution (X_1, X_2) is completely integrable. Therefore we can choose an integral submanifold N and a positive number ε so that $W \subset U$. Rewrite (17) and (18) as

$$(17)' \qquad X_1^* = fX_1 - hX_2 ,$$

$$(18)' \qquad X_2^* = hX_1 + fX_2 .$$

Then $X_1 f = X_1 h = X_2 f = X_2 h = 0$. Next we have $X_3 f = -ah$, because $(X_3 f)_{\phi_t x} = d(\cos at)/dt = -a \sin at = -ah_{\phi_t x}$. Similarly $X_3 h = af$. Then by (16) we can prove (19).

Proof of Theorem C. By Lemma 2.6 the given orthonormal frame (X_i) on U can be changed to an orthonormal frame (X_i^*) satisfying (19) with respect to g on W. By Theorem B, (W, g) is locally flat. Since for each point p of U we can choose $W(p)$, (U, g) is locally flat.

Proof of Theorem C.* Lemma 2.5 and Theorem C give a proof of Theorem C*.

3. Tangent bundles of Kählerian manifolds and proof of Theorem D

Let (M, g) be an m-dimensional Riemannian manifold, and TM its tangent bundle ($\pi: TM \to M$). For a coordinate neighborhood $(U, x^i, i = 1, \cdots, \dim M = m)$ in M we have the corresponding natural coordinate neighborhood $(\pi^{-1}U, x^i, y^i)$ in TM, where $(x^i, y^i) = y^r \partial/\partial x^r$. By $(x^i, y^i; V^i, W^i)$ we denote the vector field on TM (or the tangent vector) such that $V^r \partial/\partial x^r + W^r \partial/\partial y^r$. Let Γ^i_{jk} be the Christoffel symbols of (M, g). Then the geodesic flow vector field F is given by

$$(20) \qquad F = (x^i, y^i; y^i, -\Gamma^i_{rs} y^r y^s) .$$

Let $X = (X^i)$ be a vector field on M (or a tangent vector). Then we define vector fields on TM (or tangent vectors at (x, y)) X^* and X^v by

$$(21) \qquad X^* = (x^i, y^i; X^i, -\Gamma^i_{rs} y^r y^s) ,$$

$$(22) \qquad X^v = (x^i, y^i; 0, X^i) .$$

The Sasaki metric g^S on TM is characterized by

$$g^S(X^*, Y^*) = g(X, Y)\cdot\pi\ , \quad g^S(X^*, Y^v) = 0\ , \tag{23}$$
$$g^S(X^v, Y^v) = g(X, Y)\cdot\pi$$

for all vector fields X, Y on M (or tangent vectors at each point). Let $A = (A^i_j)$ be a $(1, 1)$-tensor field on M, and define vector fields A^* and A^v on TM by (cf. [5], [6])

$$A^* = (x^i, y^i\,; A^i_r y^r, -\Gamma^i_{ru} A^u_s y^r y^s)\ , \tag{24}$$

$$A^v = (x^i, y^i\,; 0, A^i_r y^r)\ . \tag{25}$$

Denote the 0-section in TM by (M). Let (M, g, J) be a Kählerian manifold with an almost complex structure tensor J and a Kählerian metric g. Then $TM - (M)$ admits a 3-dimensional distribution $D = (F, J^*, J^v)$. F depends on g, J^v on J, and J^* on g and J. Therefore D reflects geometric property of (M, g, J) in the tangent bundle TM.

The normal vector to each tangent sphere bundle $T_u M$ is given by $N_{(x,y)} = (x^i, y^i\,; 0, y^i)$. We see that J^v, J^*, F, N are orthogonal, since

$$J^v_{(x,y)} = (Jy)^v_{(x,y)}\ , \qquad J^*_{(x,y)} = (Jy)^*_{(x,y)}\ ,$$
$$F_{(x,y)} = (y)^*_{(x,y)}\ , \qquad N_{(x,y)} = (y)^v_{(x,y)}\ .$$

Therefore, J^v, J^* and F are tangent to each $T_u M$.

Lemma 3.1. *For J^v, F, J^* we have*

$$[J^v, F] = J^*\ , \tag{26}$$

$$[F, J^*] = (x^i, y^i\,; 0, R^i{}_{rks} J^k_t y^r y^s y^t)\ , \tag{27}$$

$$[J^*, J^v] = F\ , \tag{28}$$

where $(R^i{}_{rks}\partial/\partial x^i) = R(\partial/\partial x^k, \partial/\partial x^s)\partial/\partial x^r$.

Proof. We obtain these equations from direct calculations, using $J^i_r J^r_j = -\delta^i_j$ and $\nabla_r J^i_j = 0$. q.e.d.

A Kählerian manifold (M, g, J) is of constant holomorphic sectional curvature at x if and only if $R(X, JX)X$ is proportional to JX for any tangent vector X at x (cf. Tanno [4]). Therefore $D = (F, J^*, J^v)$ is completely integrable, if and only if $[F, J^*]$ is proportional to J^v, that is, (M, g, J) is of constant holomorphic sectional curvature at each point. In this case we have

$$[F, J^*] = Hg(y, y)J^v\ , \tag{29}$$

where $H = g(R(y, Jy)Jy, y)/g(y, y)^2$.

Proof of Theorem D. Since $\dim M = 2$, the almost complex structure tensor J (which gives the $\frac{1}{2}\pi$-rotation of tangent vectors) and g define a Kählerian structure on M. Since (M, g) is of constant curvature K, we have

$$[J^v, F] = J^* , \quad [F, J^*] = Ku^2J^v , \quad [J^*, J^v] = F ,$$

where $u^2 = g(y, y)$. Then

$$(X_1 = J^v/u,\ X_2 = F/u,\ X_3 = J^*/u)$$

is an orthonormal frame on (T_uM, g^S) and satisfies (2) with $a = Ku$, $b = c = 1/u$. Applying Theorem B* we obtain Theorem D.

Corollary 3.2. *Let $S^2(K)$ be the Euclidean 2-sphere of constant curvature K. Then $(T_uS^2(K), G^S)$, $u = 1/\sqrt{K}$, is isometric to a real projective 3-space of constant curvature $\frac{1}{4}K$.*

Proof. This follows from Theorem D and the fact that T_uS^2 is topologically a real projective space (cf. [2]).

Theorem E. *Let (M, g, J) be a Kählerian manifold of dimension ≥ 4. Then the canonical distribution $D = (F, J^*, J^v)$ on $TM - (M)$ is completely integrable if and only if (M, g, J) is of constant holomorphic sectional curvature H.*

Furthermore, $g^S([F, J^*], J^v) = Hg(y, y)^2$ holds, and hence H is positive if and only if $g^S([F, J^*], J^v)$ is positive. In this case, if (M, g) is complete, then (M, g, J) is a complex projective space with the Fubini-Study metric: $(CP^n, g, J, H) = (CP^n, H)$, $m = 2n$.

Let $L(x_0, y_0)$ be the integral submanifold of D passing through a point (x_0, y_0) of $T(CP^n, H)$ such that $g(y_0, y_0) = u^2$. Then $\pi L(x_0, y_0)$ is a complex projective line (CP^1, H), $L(x_0, y_0)$ is the tangent sphere bundle of (CP^1, H) (consisting of tangent vectors of length u), and $L(x_0, y_0)$ with the induced metric from g^S is a 3-dimensional real projective space with property (7).

We prepare two lemmas.

Lemma 3.3. *The integral curve $E_t(x_0, y_0)$ of J^v passing through a point (x_0, y_0) of TM is given by*

$$E_t(x_0, y_0) = (x_0, \cos ty_0 + \sin tJy_0) . \tag{30}$$

Proof. In a local coordinate, we have

$$\frac{dE_t(x_0, y_0)}{dt} = (x_0^i, \cos ty_0^i + \sin tJ_r^iy_0^r\,; 0, -\sin ty_0^i + \cos tJ_r^iy_0^r) ,$$

which is identical with the local expression of J^v at $E_t(x_0, y_0)$.

Lemma 3.4. *Let $L(x_0, y_0)$ be the integral submanifold of D passing through a point (x_0, y_0) of $T(CP^n, H)$. Then $\pi L(x_0, y_0) = (CP^1, H)$, and $L(x_0, y_0)$ is the tangent sphere bundle of (CP^1, H).*

Proof. Since F is the geodesic flow vector field, the projection of each integral curve of F is a geodesic in (CP^n, H). By Lemma 3.3, $L(x_0, y_0)$ contains a circle (30) in the fiber over x_0. This means that the tangent space to $\pi L(x_0, y_0)$ at x_0 is a holomorphic plane (y_0, Jy_0). All geodesics passing through x_0 and tangent to (y_0, Jy_0) define a complex projective line (CP^1, H).

Proof of Theorem E. The first part follows from the statement between the proofs of Lemma 3.1 and Theorem D, and the fact that a Kählerian manifold is of constant holomorphic sectional curvature if it is of constant holomorphic sectional curvature at each point for $m \geq 4$.

The second part follows from (29) and the well known fact that a Kählerian space form of positive holomorphic sectional curvature H is (CP^n, H).

The last part follows from Lemma 3.4 and Theorem D.

Remark. From Lemma 3.1 we see that if (M, g) is a 2-dimensional locally flat Riemannian manifold, then (T_uM, g^S) has a global orthonormal frame $(X_1 = F/u, X_2 = J^*/u, X_3 = J^v/u)$ satisfying (4) with $a = 1/u$. In particular, (T_uM, g^S) is locally flat for each u.

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